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Partition function of periodic isoradial dimer models

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Abstract Isoradial dimer models were introduced in Kenyon (Invent Math 150(2):409–439, 2002)—they consist of dimer models whose underlying graph satisfies a simple geometric condition, and whose weight function is chosen accordingly. In this paper, we prove a conjecture of (Kenyon in Invent Math 150(2):409–439, 2002), namely that for periodic isoradial dimer models, the growth rate of the toroidal partition function has a simple explicit formula involving the local geometry of the graph only. This is a surprising feature of periodic isoradial dimer models, which does not hold in the general periodic dimer case (Kenyon et al. in Ann Math, 2006).

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1 Introduction

In this paper, we solve a conjecture of [8], namely that the growth rate of the partition function of periodic isoradial dimer models can be expressed explicitly, using only the local geometry of the underlying graph. This is a surprising feature of isoradial dimer models, which does not hold in the case of general periodic dimer models. Indeed, in the general periodic case, Kenyon et al. [10] obtain an expression for the growth rate of the partition function involving elliptic integrals, but these integrals are hard to compute with explicitly. As a special case of our result, we obtain Kasteleyn's celebrated result for the growth

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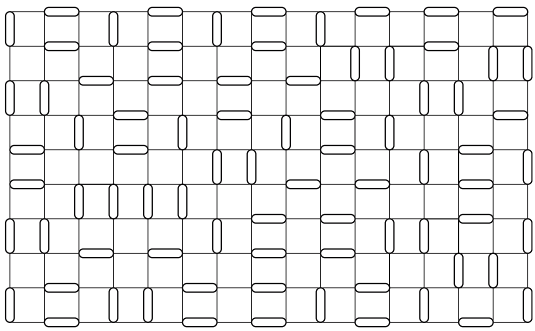
rate of the dimer partition function of the quadratic lattice $\mathbb{Z}^2$ [5,6], see Sect. 1.5. In order to state our result, let us precisely describe the setting.

1.1 Dimer model

The *dimer model* belongs to the field of statistical mechanics, and represents the adsorption of diatomic molecules on the surface of a crystal, it is defined in the following way. The surface of the crystal is modeled by a graph G. We assume that G is simple, infinite, simply connected (i.e. it is the one skeleton of a simply connected union of faces) and that its vertices are of degree ≥ 3. A *dimer configuration* of G is a *perfect matching* of G, that is a subset of edges M, such that every vertex of G touches a unique edge of M; refer to Fig. 1 for an example in the case where G is a finite subgraph of $\mathbb{Z}^2$. Let us denote by $\mathcal{M}(G)$ the set of perfect matchings of G.

Fig. 1 A dimer configuration of a finite subgraph of $\mathbb{Z}^2$



Consider a simply connected finite subgraph G_0 of G. Then, dimer configurations of G_0 are chosen with respect to the *Boltzmann measure*, defined in the following way. Assume that a positive weight function ν is assigned to edges of G, that is, each edge e of G has a weight $\nu(e)$. Then, each dimer configuration M of G_0 has an energy $\mathcal{E}(M) = -\sum_{e\in M}\log\nu(e)$. The probability of occurrence of the dimer configuration M chosen with respect to the Boltzmann measure μ_0, is given by:

$$\mu_0(M) = \frac{e^{-\mathcal{E}(M)}}{Z(G_0,\nu)} = \frac{\prod_{e\in M}\nu(e)}{Z(G_0,\nu)},$$

where $Z(G_0,\nu) = \sum_{M\in\mathcal{M}(G_0)}\prod_{e\in M}\nu(e)$ is the normalizing constant, known as the *partition function*.

The first step in the study of the dimer model is the computation of the partition function. Indeed, the latter yields precious information about the global behavior of the system [1,2,10], whose understanding is the goal of statistical mechanics. Kasteleyn [5,6], and independently Temperley and Fisher [12], laid the ground stone for the study of the partition function. Consider G_0 as above. Then they give an explicit formula for $Z(G_0,\nu)$ as the square root of the determinant of a matrix K_0, also known as a *Kasteleyn matrix*. Loosely stated, K_0 is the weighted adjacency matrix of G_0 (weighted by the function ν), where minus signs are added to coefficients in a suitable way, see Sect. 2.2 for details.

Since edges of dimer configurations represent diatomic molecules, the next step is to understand the partition function of infinite graphs. The relevant question then, is the computation of its growth rate. More precisely, if $\{G_n\}$ is an exhaustion of G by finite graphs, the goal is to compute:

$$C = \lim_{n\to\infty} \frac{1}{n^2} \log Z(G_n, \nu). \tag{1}$$

Note that this limit is not universal, in the sense that it strongly depends on the choice of exhaustion $\{G_n\}$. Moreover, in the case of exhaustions by planar graphs, this limit is hard to compute because of the lack of symmetry which reflects in the Kasteleyn matrix. This is the reason why computations are done on toroidal graphs, as explained in the next section.

1.2 Toroidal dimer models

Consider an infinite graph G as above, and suppose that G is bipartite, i.e. that it admits a bipartite coloring of its vertices. Assume moreover that there exists a bi-dimensional lattice Λ, such that G is doubly Λ-periodic, that is the graph G as well as its vertex coloring are invariant under translation by elements of Λ. Define G_n to be the toroidal graph $G/\Lambda n$, then $\{G_n\}$ is a natural exhaustion of the graph G.

In the case where G is the square lattice $\mathbb{Z}^2$, Kasteleyn [5,6] gives an explicit expression for $Z(G_n, \nu)$ as a linear combination of the determinants of four Kasteleyn matrices. Tesler [13] generalizes this result to any graph G satisfying the above, see Sect. 2.2 for details (he actually generalizes it to graphs embedded on genus g surfaces). In [10], Kenyon et al. give an explicit expression for the limit C of Eq. (1) in the case of the natural toroidal exhaustion $\{G_n\}$ of G. For general periodic, bipartite graphs G, this limit depends on the combinatorics of the model and involves elliptic integrals.

In the next section, we define isoradial dimer models—the sub-family of dimer models for which the expression for the growth rate C of (1) becomes surprisingly simple.

1.3 Isoradial dimer models

Isoradial dimer models were introduced in [8]. They are dimer models on graphs G satisfying a geometric condition called *isoradiality*. This notion first appeared in [4], see also [11], and is defined as follows. A graph G is *isoradial* if it can be embedded in the plane so that all faces of G are inscribed in a circle, and all circumcircles have the same radius, moreover, all circumcenters of the faces are contained in the closure of the faces. The common radius is taken to be 1. By a slight abuse of notation, let us denote by G an isoradial graph, as well as its isoradial embedding. An isoradial embedding of the dual graph G^* is obtained by sending dual vertices to the center of the corresponding faces.

Recall that the energy of a dimer configuration depends on the weights assigned to edges of G. In the case of isoradial graphs, one considers a specific weight function ν, called the *critical weight function*, defined as follows [8]. To each edge e of G corresponds a unit side-length rhombus $R(e)$ whose vertices are the vertices of e and of its dual edge ($R(e)$ may be degenerate). Let $\widetilde{R} = \cup_{e \in G} R(e)$. Then, define $\nu(e) = 2\sin\theta$, where 2θ is the angle of the rhombus $R(e)$ at the vertex it has in common with e; θ is called the *rhombus angle* of the edge e. Note that $\nu(e)$ is the length of e^* the dual edge of e, in the isoradial embedding of G^*.

When the graph considered for a dimer model is isoradial, and when the energy of configurations is determined by the critical weight function, we speak of an *isoradial dimer model*.

1.4 Result

Consider an isoradial dimer model. Assume moreover that G is bipartite and doubly Λ-periodic. As above consider the natural exhaustion $\{G_n\}$ of G by toroidal graphs. Then, Theorem 1 below gives an explicit expression for the growth rate of the partition function of the exhaustion $\{G_n\}$ as a function of the local geometry of the graph.

Theorem 1

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z(G_n, \nu) = \sum_{i=1}^{m} \left(\frac{\theta_i}{\pi} \log 2 \sin\theta_i + \frac{1}{\pi} L(\theta_i) \right), \tag{2}$$

where, $\theta_1, \ldots, \theta_m$ are the rhombus angles of the edges $e_1, \ldots, e_m$ of G_1, and L is Lobachevsky's function, $L(x) = -\int_0^x \log 2 \sin t \, dt$.

- The right hand side of (2) is (up to a factor 2), the expression Kenyon obtained for what he calls the *log of the normalized determinant* of the Dirac operator [8]. He conjectured it to be the right limit for the growth rate of the partition function of an appropriate exhaustion of G. Theorem 1 states that this is the right limit in the case of the natural exhaustion $\{G_n\}$ of G by toroidal graphs.
- The surprising feature of Theorem 1 is that the growth rate of the partition function is expressed using geometric information of the fundamental domain G_1 only, no combinatorics is involved. This is in contrast with the result of [10] for general bipartite periodic graphs. Note that, although the result of [10] is true in a more general setting, there seems to be no simple way of re-deriving Theorem 1 from it. Let us also mention that we use the result of [10] in order to have a priori existence of the limit C of Eq. (1).

1.5 Example: Kasteleyn's computation

In [5,6], Kasteleyn skillfully computes the toroidal partition function for $G = \mathbb{Z}^2$. As a special case of Theorem 1, let us rederive his result in a two line calculation. Note that the following computation was already mentioned in [8], as a support for the conjecture proved in Theorem 1. When $G = \mathbb{Z}^2$, the critical weight function is $\nu \equiv \sqrt{2}$, see Fig. 2 (left). Moreover, consider the fundamental domain $G_1 = G/\Lambda$ as in Fig. 2 (right).

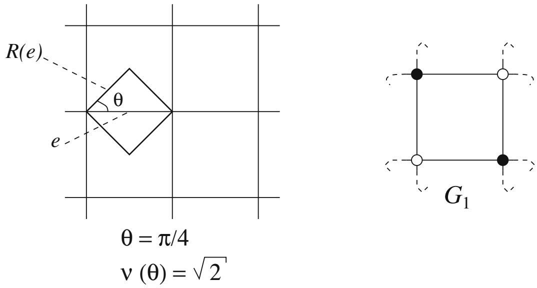


Fig. 2 Critical weight function for $\mathbb{Z}^2$ (*left*). Fundamental domain G_1 (*right*)

Then, a direct computation using Theorem 1 yields:

$$\lim_{n\to\infty} \frac{1}{2n^2} \log Z(G_n, \sqrt{2}) = 4\left(\frac{1}{4}\log\sqrt{2} + \frac{1}{\pi}L\left(\frac{\pi}{4}\right)\right), \\ = \frac{1}{2}\log 2 + \frac{2\mathcal{K}}{\pi},$$

where $\mathcal{K}$ is Catalan's constant. Note that the $\frac{1}{2}\log 2$ factor differs from Kasteleyn's computation, because we have weights $\sqrt{2}$ on the edges instead of weights 1.

2 Proof of Theorem 1

In the whole of this section, we let G be an infinite, bipartite, Λ-periodic, isoradial graph. We assume that edges of G are assigned the critical weight function. Moreover, W denotes the set of white vertices of G, B the set of black ones, and G_n the toroidal graph $G/\Lambda n$.

Before giving the proof of Theorem 1, let us precisely describe the conjecture of [8]. In [8], Kenyon introduces the Dirac operator K indexed by vertices of G (we refer the reader to [8] for the definition of K, and to Sect. 2.1 for the definition of the *real Dirac operator* K which is closely related). He also defines its inverse K^{-1}, for which he proves existence and uniqueness. One of his beautiful results is an explicit expression for the *log of the normalized determinant* "$\log \det_1 K$" of the Dirac operator K, defined by the following PDE:

$$\frac{\partial \log \det_1 K}{\partial K(w,b)} = \frac{1}{|V(G_1)|} K^{-1}(b,w), \tag{3}$$

where $w \in W$, $b \in B$ are adjacent vertices, $|V(G_1)|$ is the number of vertices of G_1. Adding some initial condition allows him to solve the PDE, and obtain the explicit expression for $\log \det_1 K$, which is the right hand side of Eq. (2). How is this related to the dimer model? The Dirac operator K is closely related to an infinite Kasteleyn matrix—to have an actual infinite Kasteleyn matrix, one needs to work with the *real Dirac operator* K (see Sect. 2.1 below), obtained from the Dirac operator by a gauge transformation. If G_0 is a finite subgraph of G, and K_0 the restriction of K to the vertices of G_0, then $\log \det_1 \mathsf{K}_0$ can be defined naturally as $\frac{1}{|V(G_0)|} \log \det \mathsf{K}_0$, which is the normalized log of the dimer partition function of G_0. Our approach to solving Kenyon's conjecture is to interpret (3) as the limit of the dimer partition function on some appropriate exhaustion of G. The problem lies in proving the existence of such limits, leading us to work with toroidal exhaustions. Note that part of the proof is close to [8], since it consists in solving the PDE (3) for operators restricted to subgraphs of the exhaustion.

The structure of Sect. 2 is as follows. Sect. 2.1 consists in the definition of the real Dirac operator K and its inverse K^{-1}. The operator K was introduced in [3], see also [9]. Note that features of K are closely related to those of the Dirac operator K of [8]. In Sect. 2.2, we state the explicit expression of [6,13] for the toroidal partition function. Section 2.3 uses Sects. 2.1 and 2.2, and consists in the proof of Theorem 1.

2.1 Real Dirac operator

The *real Dirac operator* K is obtained from the *Dirac operator K* of [8] by a gauge transformation. Both operators are represented by weighted infinite adjacency matrices indexed by vertices of G. For K the edges are unoriented and weighted by the critical weight function times a complex number of modulus 1. For K, edges are oriented with a *clockwise odd* orientation (see below), and are weighted by the critical weight function. Both weight functions yield the same Boltzmann measure on finite simply connected sub-graphs of G, but, and this is the reason why the real Dirac operator K is introduced, these weights do not yield the same probability distribution on toroidal subgraphs. Note that the real Dirac operator was already used in [3], see also [9].

2.1.1 Definition

Following Kasteleyn [6], let us define clockwise odd orientations on edges of G. An elementary cycle $\mathcal{C}$ of G is said to be *clockwise odd* if, when traveling cw (clockwise) around the edges of $\mathcal{C}$, the number of co-oriented edges is odd. Note that since G is bipartite, the number of contra-oriented edges is also odd. Then, an orientation of the edges of G is defined to be *clockwise odd* if all elementary cycles are clockwise odd. In [6], Kasteleyn shows that, for planar simply connected graphs, such an orientation always exists.

Consider a clockwise odd orientation of the edges of G. Define K to be the infinite adjacency matrix of the graph G, weighted by the critical weight function ν. That is, if v_1 and v_2 are not adjacent, $\mathsf{K}(v_1, v_2) = 0$. If $w \in W$ and $b \in B$ are adjacent vertices, then $\mathsf{K}(w, b) = -\mathsf{K}(b, w) = (-1)^{\mathbb{I}_{(w,b)}} \nu(wb)$, where $\mathbb{I}_{(w,b)} = 0$ if the edge wb is oriented from w to b, and 1 if it is oriented from b to w. The infinite matrix K defines the *real Dirac operator* $\mathsf{K}\colon \mathbb{C}^{V(G)} \to \mathbb{C}^{V(G)}$, by

$$(\mathsf{K}f)(v) = \sum_{u \in G} \mathsf{K}(v, u) f(u).$$

The matrix K is also called a *Kasteleyn matrix* for the underlying dimer model.

2.1.2 Inverse real Dirac operator

Similarly to the definition of the inverse Dirac operator K^{-1} [8], the *inverse real Dirac operator* K^{-1} is defined to be the unique operator satisfying:

1. $\mathsf{K}\mathsf{K}^{-1} = \mathrm{Id}$,
2. $\mathsf{K}^{-1}(b, w) \to 0$, when $|b - w| \to \infty$.

The rational functions $\mathsf{f}_{wx}(z)$ are the analog of the rational functions $f_{wv}(z)$ of [8], and are defined in the following way. Let $w \in W$, and let $x \in B$ (resp. $x \in W$); consider the edge-path $w = w_1, b_1, \ldots, w_k, b_k = x$ (resp. $w = w_1, b_1, \ldots, w_k, b_k, w_{k+1} = x$) of G^* from w to x. Let $R(w_j b_j)$ be the rhombus associated to the edge $w_j b_j$, and denote by w_j, x_j, b_j, y_j its vertices in cclw (counterclockwise) order; $\mathrm{e}^{\mathrm{i}\alpha_j}$ is the complex vector $y_j - w_j$, and $\mathrm{e}^{\mathrm{i}\beta_j}$ is the complex vector $x_j - w_j$. In a similar way, denote by w_{j+1}, x'_j, b_j, y'_j the vertices of the rhombus $R(w_{j+1} b_j)$ in cclw order, then $\mathrm{e}^{\mathrm{i}\alpha'_j}$ is the complex vector $y'_j - w_{j+1}$, and $\mathrm{e}^{\mathrm{i}\beta'_j}$ is the complex vector $x'_j - w_{j+1}$. The function $\mathsf{f}_{wx}(z)$ is defined inductively along the path,

$$\begin{aligned}
\mathsf{f}_{ww}(z) &= 1, \\
\mathsf{f}_{wb_j}(z) &= \mathsf{f}_{ww_j}(z) \frac{(-1)^{\mathbb{I}_{(w_j, b_j)}} \mathrm{e}^{\mathrm{i}\frac{\alpha_j + \beta_j}{2}}}{\left(z - \mathrm{e}^{\mathrm{i}\alpha_j}\right)\left(z - \mathrm{e}^{\mathrm{i}\beta_j}\right)}, \\
\mathsf{f}_{ww_{j+1}}(z) &= \mathsf{f}_{wb_j}(z) (-1)^{\mathbb{I}_{(w_{j+1}, b_j)}} \mathrm{e}^{-\mathrm{i}\frac{\alpha'_j + \beta'_j}{2}} \left(z - \mathrm{e}^{\mathrm{i}\alpha'_j}\right)\left(z - \mathrm{e}^{\mathrm{i}\beta'_j}\right).
\end{aligned}$$

By Lemma 9 of [3], the function f_{wx} is well defined. Moreover, Lemma 10 of [3] gives an explicit expression for K^{-1} (it is the analog of Theorem 4.2 of [8]):

Lemma 2 [3] *The inverse real Dirac operator is given by*

$$\mathsf{K}^{-1}(b,w)=\frac{1}{4\pi^2 i}\int\limits_C \mathsf{f}_{wb}(z)\log z\,\mathrm{d}z, \tag{4}$$

where C is a closed contour surrounding cclw the part of the circle $\{\mathrm{e}^{\mathrm{i}\theta}\,|\,\theta\in[\theta_0-\pi+\Delta,\theta_0+\pi-\Delta]\}$, which contains all the poles of f_{wb}, and with the origin in its exterior.

2.2 Partition function of the torus

In this section, we give the explicit formula of Kasteleyn [6] (in the $\mathbb{Z}^2$ case), and Tesler [13] (in the general periodic case) for the toroidal partition function $Z(G_n,\nu)$. Note that this result was already described in [3], we nevertheless choose to repeat it here since its understanding is important for the proof of Theorem 1.

Let us first orient the edges of G. Consider the graph G_1, it is a bipartite graph on the torus. Fix a reference matching M_0 of G_1. For every other perfect matching M of G_1, consider the superposition $M\cup M_0$ of M and M_0, then $M\cup M_0$ consists of doubled edges and cycles. Let us define four parity classes for perfect matchings M of G_1: (e,e) consists of perfect matchings M, for which cycles of $M\cup M_0$ circle the torus an even number of times horizontally and vertically; (e,o) consists of perfect matchings M, for which cycles of $M\cup M_0$ circle the torus and even number of times horizontally, and an odd number of times vertically; (o,e) and (o,o) are defined in a similar way. By Tesler [13], one can construct an orientation of the edges of G_1, so that the corresponding adjacency matrix K_1^1 has the following property: perfect matchings which belong to the same parity class have the same sign in the expansion of the determinant; moreover of the four parity classes, three have the same sign and one the opposite sign. By an appropriate choice of sign, we can make the (e,e) class have the plus sign in $\det\mathsf{K}_1^1$, and the other three have minus sign. Consider a horizontal and a vertical cycle of the dual graph G_1^* of G_1. Then define K_2^1 (resp. K_3^1) to be the matrix K_1^1 where the sign of the coefficients corresponding to edges crossing the horizontal (resp. vertical) cycle is reversed; and define K_4^1 to be the matrix K_1^1 where the sign of the coefficients corresponding to the edges crossing both cycles are reversed. By Kasteleyn [6] (in the $\mathbb{Z}^2$ case), and Tesler [13] (in the general case), we have the following,

$$Z(G_1,\nu)=\frac{1}{2}\left(-\det\mathsf{K}_1^1+\det\mathsf{K}_2^1+\det\mathsf{K}_3^1+\det\mathsf{K}_4^1\right).$$

The orientation of the edges of G_1 defines a periodic orientation of the graph G. For every n, consider the graph G_n, and the four matrices $\mathsf{K}_1^n,\mathsf{K}_2^n,\mathsf{K}_3^n,\mathsf{K}_4^n$ defined as above. These matrices are called the *Kasteleyn matrices* of the graph G_n.

Theorem 3 [6,13]

$$Z(G_n,\nu)=\frac{1}{2}\left(-\det \mathsf{K}_1^n+\det \mathsf{K}_2^n+\det \mathsf{K}_3^n+\det \mathsf{K}_4^n\right).$$

The orientation defined on the edges of the graph G is a clockwise odd orientation. Let K be the real Dirac operator indexed by the vertices of G, corresponding to this clockwise odd orientation. Note that except for edges crossing the horizontal and the vertical cycle, the coefficients of the Kasteleyn matrices K_ℓ^n and of the real Dirac operator agree on edges they have in common.

2.3 Proof

Consider an orientation of the edges of G defined as in Sect. 2.2, and let K be the real Dirac operator indexed by vertices of G, corresponding to this orientation. Let $\mathsf{K}_1^n,\mathsf{K}_2^n,\mathsf{K}_3^n,\mathsf{K}_4^n$ be the Kasteleyn matrices of the graph G_n. The following computations are inspired by [8]. By linear algebra, for every $\ell=1,\ldots,4$, and for every edge w_ib_i of G_n, we have

$$\frac{\partial \det \mathsf{K}_\ell^n}{\partial\left(\mathsf{K}_\ell^n\,(w_i,b_i)\right)}=\left(\det \mathsf{K}_\ell^n\right)\left(\left(\mathsf{K}_\ell^n\right)^{-1}(b_i,w_i)\right).$$

Denote by $e_1=w_1b_1,\ldots,e_m=w_mb_m$ the edges of G_1, and let $\theta_1,\ldots,\theta_m$ be the corresponding rhombus angles. Since the graph G_n is invariant under Λ-translates, we know that for every Λ-translate $w_i^tb_i^t$ of the edge w_ib_i, the coefficient $\mathsf{K}_\ell^n(w_i^t,b_i^t)=\pm\mathsf{K}_\ell^n(w_i,b_i)$. The minus sign only occurs when $\ell=2,3,4$, (recall that in the definition of $\mathsf{K}_2^n,\ldots,\mathsf{K}_4^n$, the sign of the entries which cross the horizontal and/or the vertical cycle of G_n^* is reversed), but then $(\mathsf{K}_\ell^n)^{-1}(b_i^t,w_i^t)=\pm(\mathsf{K}_\ell^n)^{-1}(b_i,w_i)$. Hence, for every $\ell=1,\ldots,4$, and for every $i=1,\ldots,m$,

$$\begin{aligned}\frac{\partial \det \mathsf{K}_\ell^n}{\partial\theta_i}&=\sum_{w_i^tb_i^t\text{ translates of }w_ib_i}\frac{\partial \det \mathsf{K}_\ell^n}{\partial\left(\mathsf{K}_\ell^n\left(w_i^t,b_i^t\right)\right)}\frac{\partial\left(\mathsf{K}_\ell^n\left(w_i^t,b_i^t\right)\right)}{\partial\theta_i},\\&=n^2\left(\det \mathsf{K}_\ell^n\right)\left(\left(\mathsf{K}_\ell^n\right)^{-1}(b_i,w_i)\right)\frac{\partial\mathsf{K}_\ell^n(w_i,b_i)}{\partial\theta_i},\end{aligned}\tag{5}$$

where the edges $e_1,\ldots,e_m$ do not cross the horizontal and the vertical path of the dual graph G_n^* of G_n. Define the function $\varphi_n(\theta_1,\ldots,\theta_m)$ by

$$\varphi_n(\theta_1,\ldots,\theta_m)=\frac{1}{n^2}\log Z(G_n,\nu).$$

By Theorem 3, we have $Z(G_n,\nu)=\frac{1}{2}\left(-\det \mathsf{K}_1^n+\det \mathsf{K}_2^n+\det \mathsf{K}_3^n+\det \mathsf{K}_4^n\right)$. Moreover, for every $\ell=1,\ldots,4$, $\mathsf{K}_\ell^n(w_i,b_i)=\mathsf{K}(w_i,b_i)$, so that using (5), we obtain for every $i=1,\ldots,m$,

$$\frac{\partial \varphi_n}{\partial \theta_i}(\theta_1,\ldots,\theta_m) = \left(\frac{\partial \mathsf{K}(w_i,b_i)}{\partial \theta_i}\right) \times \left(\frac{-\det \mathsf{K}_1^n}{2Z(G_n,\nu)}\left(\mathsf{K}_1^n\right)^{-1}(b_i,w_i) + \sum_{\ell=2}^{4} \frac{\det \mathsf{K}_\ell^n}{2Z(G_n,\nu)}\left(\mathsf{K}_\ell^n\right)^{-1}(b_i,w_i)\right). \tag{6}$$

The next part of the argument can be found in [7]. The second bracket of Eq. (6) is a weighted average of the four quantities $(\mathsf{K}_\ell^n)^{-1}(b_i,w_i)$, with weights $\pm\frac{1}{2}\det \mathsf{K}_\ell^n/Z(G_n,\nu)$. These weights are all in the interval $(-1/2,1/2)$ since for every $\ell = 1,\ldots,4$, $2Z(G_n,\nu) > |\det \mathsf{K}_\ell^n|$. Indeed, $Z(G_n,\nu)$ counts the weighted sum of dimer configurations of G_n, whereas $|\det \mathsf{K}_\ell^n|$ counts some configurations with negative sign. Since the weights sum to 1, the weighted average converges to the same value as each $\left(\mathsf{K}_\ell^n\right)^{-1}(b_i,w_i)$.

By Proposition 2 of [3], for every $\ell = 1,\ldots,4$, $(\mathsf{K}_\ell^n)^{-1}(b_i,w_i)$ converges to $\mathsf{K}^{-1}(b_i,w_i)$ on a subsequence (n_j) of n's. Hence, for every $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, $n \in (n_j)$, Eq. (6) can be written as

$$\frac{\partial \varphi_n}{\partial \theta_i}(\theta_1,\ldots,\theta_m) = \frac{\partial \mathsf{K}(w_i,b_i)}{\partial \theta_i}\mathsf{K}^{-1}(b_i,w_i) + \varepsilon. \tag{7}$$

Let us compute the right hand side of (7), using the notations of Sect. 2.1.2. By definition, $\mathsf{K}(w_i,b_i) = (-1)^{\mathbb{I}_{(w_i,b_i)}}2\sin\theta_i$, hence

$$\frac{\partial \mathsf{K}(w_i,b_i)}{\partial \theta_i} = (-1)^{\mathbb{I}_{(w_i,b_i)}}2\cos\theta_i. \tag{8}$$

Moreover, by definition $\mathsf{f}_{w_ib_i}(z) = (-1)^{\mathbb{I}_{(w_i,b_i)}}\frac{e^{i\frac{(\alpha_i+\beta_i)}{2}}}{(z-e^{i\alpha_i})(z-e^{i\beta_i})}$. So that using Lemma 2, and the Residue Theorem yields:

$$\begin{aligned}
\mathsf{K}^{-1}(b_i,w_i) &= \frac{1}{4\pi^2 i}(-1)^{\mathbb{I}_{(w_i,b_i)}}e^{i\frac{(\alpha_i+\beta_i)}{2}}\int_C \frac{\log z}{(z-e^{i\alpha_i})(z-e^{i\beta_i})}dz,\\
&= \frac{1}{2\pi}(-1)^{\mathbb{I}_{(w_i,b_i)}}e^{i\frac{(\alpha_i+\beta_i)}{2}}\left(\frac{i\alpha_i}{e^{i\alpha_i}-e^{i\beta_i}} + \frac{i\beta_i}{e^{i\beta_i-\alpha_i}}\right),\\
&= \frac{1}{2\pi}(-1)^{\mathbb{I}_{(w_i,b_i)}}\frac{\theta_i}{\sin\theta_i}.
\end{aligned} \tag{9}$$

Combining Eqs. (8) and (9), implies:

$$\frac{\partial \varphi_n}{\partial \theta_i}(\theta_1,\ldots,\theta_m) = \frac{\theta_i}{\pi}\mathrm{cotan}\,\theta_i + \varepsilon.$$

By [8], there is a continuous way to deform the graph G so that all rhombus angles tend to 0 or $\pi/2$. The same transformation can be applied to G_n, for every

n. Denote by θ_i^0 the angle θ_i after such a deformation. Let M be the number of angles $\theta_1^0,\dots,\theta_m^0$ which are equal to $\pi/2$. By the argument of [8], for $n \geq n_0$, $n \in (n_j)$, we obtain

$$\varphi_n(\theta_1,\dots,\theta_m) = \sum_{i=1}^{m} \left(\frac{\theta_i}{\pi} \log 2\sin\theta_i + \frac{1}{\pi} L(\theta_i) \right) - \frac{M}{2}\log 2 + \varphi_n(\theta_1^0,\dots,\theta_m^0) + \varepsilon. \tag{10}$$

Let us compute $\varphi_n(\theta_1^0,\dots,\theta_m^0)$. Suppose the above deformation is applied to the graph G, then edges corresponding to rhombus angles 0 have weight 0, and those corresponding to rhombus angles $\pi/2$ have weight 2. Removing the 0 weight edges, the deformed graph consists of independent copies of $\mathbb{Z}$. Applying the deformation to G_1, we obtain graphs $\mathbb{Z}/m_1\mathbb{Z},\dots,\mathbb{Z}/m_p\mathbb{Z}$, for even $m_1,\dots,m_p$, where each $\mathbb{Z}/m_j\mathbb{Z}$ consists of weight 2 edges. We can compute,

$$Z(G_n,\nu) = (2^{\frac{nm_1}{2}}2)^n \cdots (2^{\frac{nm_p}{2}}2)^n = 2^{n^2\frac{M}{2}}2^{np}.$$

Hence, for every $\varepsilon > 0$, there exists n_1, such that for every $n \geq n_1$, we have

$$\varphi_n(\theta_1^0,\dots,\theta_m^0) = \frac{M}{2}\log 2 + \varepsilon. \tag{11}$$

Combining Eqs. (10) and (11) yields, for every $n \geq \max\{n_0, n_1\}$, $n \in (n_j)$,

$$\varphi_n(\theta_1,\dots,\theta_m) = \sum_{i=1}^{m} \left(\frac{\theta_i}{\pi} \log 2\sin\theta_i + \frac{1}{\pi} L(\theta_i) \right) + 2\varepsilon.$$

This implies,

$$\lim{}'_{n\to\infty}\varphi_n(\theta_1,\dots,\theta_m) = \sum_{i=1}^{m} \left(\frac{\theta_i}{\pi} \log 2\sin\theta_i + \frac{1}{\pi} L(\theta_i) \right),$$

where the limit is taken on the subsequence (n_j). Using Theorem 3.5 of [10], we deduce that $\varphi_n(\theta_1,\dots,\theta_m)$ converges (as $n \to \infty$) to the same limit as the above subsequence. □

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